HOMEWORK #3 SELECTED SOLUTIONS

1.6: #1d

<u>Claim</u>: There do not exist $m, n \in \mathbb{Z}$ such that

$$12m + 15n = 1.$$

<u>Proof</u>: Suppose $m, n \in \mathbb{Z}$ with $12m + 15n = 1$.

Then $$3(4m + 5n) = 1$$

and since $4m + 5n \in \mathbb{Z}$ this means $3 \mid 1$, which is false. ∎

1.6: #3

<u>Claim</u>: If every even natural number greater than 2 is the sum of two primes then every odd natural number greater than 5 is the sum of three primes.

<u>Proof</u>: Suppose every even number greater than 2 is the sum of two primes and let $n \in \mathbb{Z}$ be odd with $n > 5$.

This means $\exists\, k \in \mathbb{Z}$ with $n = 2k+1$ so

$$n - 3 = (2k+1) - 3 = 2(k-1)$$

is even and $n > 5 \Rightarrow n - 3 > 2$.

Thus $n-3$ is an even integer greater than 2 so by assumption

$$n-3 = p_1 + p_2$$

where $p_1, p_2 \in \mathbb{Z}$ are prime.

Then $$n = p_1 + p_2 + 3$$

so n is the sum of three primes since 3 is a prime. $\blacksquare$

$\boxed{1.7{:}5b}$

<u>Claim</u>: If $x \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$, then $x+y \in \mathbb{R} - \mathbb{Q}$.

<u>Proof</u>: Let $x \in \mathbb{Q}$, $y \in \mathbb{R} - \mathbb{Q}$ and assume $x+y \in \mathbb{Q}$.

Since $x, x+y \in \mathbb{Q}$ by def $\exists\, a, b, c, d \in \mathbb{Z}$ with $x = \frac{a}{b}$, $x+y = \frac{c}{d}$ and $b, d \neq 0$.

Then $$x+y = \frac{c}{d} \Rightarrow \frac{a}{b} + y = \frac{c}{d}$$
$$\Rightarrow y = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{db}$$

so $y \in \mathbb{Q}$, since $cd-ad, db \in \mathbb{Z}$ and

$db \neq 0$ since $d, b \neq 0$. This contradicts our assumption that $y \notin \mathbb{Q}$. $\blacksquare$

$\boxed{1.7: \# 5c}$

Claim: $\exists$ irrational x, y such that $x+y \in \mathbb{Q}$.

Pf: Let x be an irrational number, so $x \in \mathbb{R} - \mathbb{Q}$ (for instance could take $x = \sqrt{2}$)

and let $y = 1 - x$

Then $x + y = x + (1-x) = 1 \in \mathbb{Q}$

so the sum is rational. We must show $y \notin \mathbb{Q}$.

Note if $y \in \mathbb{Q}$ then $\exists\ p, q \in \mathbb{Z}$ $(q \neq 0)$ such that

$y = p/q$

and then $y = 1 - x \Rightarrow x = 1 - p/q$

$$\Rightarrow x = \frac{q-p}{q}$$

which is rational since $q-p, q \in \mathbb{Z}$ and $q \neq 0$.

Thus $x \in \mathbb{Q}$, a contradiction.

We conclude $y \notin \mathbb{Q}$. $\blacksquare$